RELATIVISTIC TRANSFORMATIONS FOR TIME SYNCHRONIZATION AND DISSEMINATION IN THE SOLAR SYSTEM

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Abstract

The measurement of time is an essential aspect of navigation. The Global Positioning System (GPS) is comprised of a constellation of satellites that transmit one-way pseudorandom noise (PRN) coded signals used for range and time measurements. The signals are referenced to onboard atomic clocks. The GPS provides a model for position determination with a precision of a few meters and time dissemination with a precision of about 10 nanoseconds. The mathematical algorithms used in the GPS receiver require the application of the principles of general relativity. Similar models will be needed for high-precision navigation in the solar system. The adoption of an appropriate coordinate system and time scale is required. This paper outlines the fundamental concepts of relativistic time transfer and describes the details of the mathematical model. The approximate magnitudes of various relativistic effects for clocks onboard the GPS satellites, other satellites in Earth orbit, and a clock on the surface of Mars or on the Moon are derived.

INTRODUCTION

Relativity has become an important aspect of modern precise timekeeping systems. Thus, far from being simply a textbook problem or merely of theoretical scientific interest, the analysis of relativistic effects on time measurement is an essential practical consideration. The Global Positioning System (GPS) is an example of an engineering system in which the recognition of appropriate relativistic corrections are necessary for its successful operation. Clocks onboard GPS satellites run fast by 38 μs/d due to their altitude and velocity and have a periodic component with amplitude of about 46 ns due to the small orbit eccentricity compared to clocks on the geoid. The constant drift is compensated by a rate offset prior to launch. Neglect of the periodic effect would result in a radial position error of about 15 m. Similarly, relativistic transformations between clocks throughout the solar system will be required in future space missions. The purpose of this paper is to describe the fundamental theoretical principles for relativistic time transfer. The principal relativistic effects are derived for time transfer between clocks on the Earth's surface to clocks on Earth-orbiting satellites, Mars, and the Moon.

PROPER TIME AND COORDINATE TIME

In the theory of general relativity, there are two kinds of time. Proper time τ is the actual reading of a clock. The proper times are different for clocks in different states of motion and in different gravitational potentials. The proper time measured by a clock may be compared to the proper time measured by another clock through the intermediate variable t called coordinate time, which, by definition, has the same value everywhere for a given event. The relationship between coordinate time and proper time depends on the velocity of the clock and the gravitational potential at the location of the clock. It is established through the invariance of the four-dimensional space-time interval.

The theory of space, time, and gravitation according to the general theory of relativity is founded upon the notion of an invariant Riemannian space-time interval of the form

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{00}\, c^2\, dt^2 + 2\sum_{j=1}^{3} g_{0j}\, c\, dt\, dx^j + \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\, dx^i\, dx^j$$

where $x^\alpha \equiv (c\,t, x^i)$. The fundamental mathematical object is the metric tensor $g_{\mu\nu}$, whose components are functions of the coordinates and are symmetric in the indices μ, ν (that is, $g_{\mu\nu} = g_{\nu\mu}$). The metric tensor plays the role of the gravitational potentials.

For a transported clock, the space-time interval is

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu \equiv -c^2 d\tau^2$$

where τ is the proper time recorded by the clock. For a given coordinate system, this equation establishes a well-defined transformation between coordinate time and proper time. The coordinate time is arbitrary, as the comparison is made between two proper times. In an inertial coordinate system with no gravitation, the metric components are $-g_{00} = 1$, $g_{11} = g_{22} = g_{33} = 1$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$. Then $ds^2 = -c^2 (1 - v^2 / c^2)\, dt^2 = -c^2\, d\tau^2$, where v is the clock velocity as in special relativity, which implies the phenomenon of time dilation of a moving clock relative to a stationary clock.

For an electromagnetic signal, the space-time interval satisfies the condition

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu = 0$$

In an inertial coordinate system with no gravitation, this reduces to $ds^2 = -c^2\, dt^2 + dx^2 + dy^2 + dz^2 = 0$ as in special relativity, whose invariance implies that the speed of light is c in all inertial systems.

EARTH-ORBITING SATELLITE CLOCKS

To a sufficient approximation in the analysis of clock transport, the components of the metric tensor in an Earth-Centered Inertial (ECI) coordinate system are $-g_{00} \approx 1 - 2\,U / c^2$, $g_{0j} = 0$, and $g_{ij} \approx \delta_{ij}$, where U is the Newtonian gravitational potential and δ_{ij} is the Kronecker delta. For a clock onboard a satellite, the elapsed coordinate time is given in terms of the proper time by the integral

$$\Delta t = \int_{\tau_0}^{\tau} \left(1 + \frac{1}{c^2} U + \frac{1}{2}\frac{1}{c^2} v^2\right) d\tau$$

The first term under the integral is the elapsed proper time, the second term is the correction due to the gravitational potential U (gravitational redshift), and the third term is the correction due to the velocity v of the satellite (time dilation).

In the rotating Earth-Centered Earth-Fixed (ECEF) coordinate system, it is convenient to apply a change of scale to define a new coordinate time

$$\Delta t' = \left(1 - \frac{1}{c^2} W_0\right) \Delta t = \int_{\tau_0}^{\tau} \left\{1 + \frac{1}{c^2}(U - W_0) + \frac{1}{2}\frac{1}{c^2} v^2\right\} d\tau$$

where $W_0 = 6.2637 \times 10^7$ m^2/s^2 is the geopotential over the surface of the Earth, which is a constant. Then the coordinate time $\Delta t'$ corresponds to the proper time registered by a clock at rest on the geoid. Thus, the clock becomes a coordinate clock. Upon integration, the elapsed coordinate time for an Earth-orbiting satellite clock is

$$\Delta t' = \left(1 + \frac{3}{2}\frac{1}{c^2}\frac{GM}{a} - \frac{1}{c^2} W_0\right) \Delta \tau + \frac{2}{c^2}\sqrt{GM\, a}\; e \sin E$$

where a is the orbital semimajor axis, e is the orbital eccentricity, and E is the eccentric anomaly. The first term represents a constant rate offset between the satellite clock and a clock on the geoid. The second term is a small relativistic periodic correction due to the orbital eccentricity. This term may be expressed without approximation as

$$\Delta t_{rel} = \frac{2}{c^2}\sqrt{GM\, a}\; e \sin E = \frac{2\,\mathbf{r}\cdot\mathbf{v}}{c^2}$$

where **r** and **v** are the position and velocity vectors of the satellite. As $\mathbf{r}\cdot\mathbf{v}$ is a scalar, it may be evaluated in either the ECI or ECEF coordinate system.

ELECTROMAGNETIC SIGNALS

The coordinate time of propagation of an electromagnetic signal is

$$\Delta t = \frac{\rho}{c} + \frac{1}{c}\int_{\text{path}} \sum_{j=1}^{3} \frac{g_{0j}}{-g_{00}}\, dx^i$$

where ρ is the propagation path length. The integral term is called the Sagnac effect. In the rotating ECEF coordinate system, the metric components are $-g_{00} \approx 1$, $g_{0j} = (\boldsymbol{\omega}\times\mathbf{r})_j / c$, and $g_{ij} \approx \delta_{ij}$, where $\boldsymbol{\omega}$ is the Earth's rotational angular velocity. Therefore, the Sagnac effect between two points A and B is

$$\Delta t_{\text{Sagnac}} \approx \frac{1}{c^2}\int_A^B (\boldsymbol{\omega}\times\mathbf{r})\cdot d\mathbf{r} = \frac{1}{c^2}\int_A^B \boldsymbol{\omega}\cdot(\mathbf{r}\times d\mathbf{r}) = 2\frac{1}{c^2}\int_A^B \boldsymbol{\omega}\cdot d\mathbf{A} = \frac{2\,\omega A}{c^2}$$

where A is the perpendicular projection of the area formed by the center of rotation and the endpoints of the light path. For endpoints at (x_A, y_A) and (x_B, y_B), the Sagnac effect may be expressed

$$\Delta t_{\text{Sagnac}} = \frac{2\,\omega A}{c^2} = \frac{\omega}{c^2}\left(x_A\, y_B - y_A\, x_B\right) .$$

In the case of a receiver at rest on the Earth, an observer in the ECEF frame regards the receiver as stationary and applies the Sagnac correction. However, an observer in the ECI frame sees that the receiver has moved due to the Earth's rotation during the signal time of flight and instead applies a propagation time correction due to the additional path length. The term "Sagnac effect" is part of the vocabulary of only the observer in the rotating reference frame. The corresponding correction applied by the inertial observer might be called a "velocity correction." While the interpretation of the correction is different in the two frames, the numerical value is the same in either frame.

RELATIVISTIC EFFECTS ON SATELLITE CLOCKS AND SIGNALS

THE GLOBAL POSITIONING SYSTEM

The GPS has served as a laboratory for doing relativity physics. The consistent application of relativity to time and position measurements has been demonstrated by the operational precision of the system and by numerous experiments designed to test the individual effects over a wide range of conditions. The GPS provides a model for the application of relativity algorithms to similar applications across a broad spectrum of timekeeping systems. The relativistic effects encountered in the GPS illustrate that the effects that must be considered are not negligible. The satellites transmit one-way pseudorandom noise (PRN) coded signals that are used for range and time measurements. The signals are referenced to onboard atomic clocks. For measurements with a precision at the 1-to-10 nanosecond level, there are three relativistic effects that must be considered.

First, there is the effect of time dilation. The velocity of a moving clock causes it to appear to run slow relative to a clock on the Earth. GPS satellites revolve around the Earth with an orbital period of 11.967 hours and a velocity of 3.874 km/s. Thus, on account of its velocity, a GPS satellite clock appears to run slow by 7 μs per day.

Second, there is the effect of the gravitational redshift, a frequency shift caused by the difference in gravitational potential. (The term "redshift" is generic, regardless of sign, but for a satellite clock the frequency shift is actually a "blueshift.") The difference in gravitational potential between the altitude of the orbit and the surface of the Earth causes the satellite clock to run fast. At an altitude of 20,184 km, the clock runs fast by 45 μs per day.

The net effect of time dilation and gravitational redshift is that the satellite clock runs fast by approximately 38 μs per day when compared to a similar clock at rest on the geoid, including the effects of the Earth's rotation and the gravitational potential at the Earth's surface. This is an enormous rate difference for a clock that maintains time with a precision of a few nanoseconds over a day. To compensate for this large secular effect, the clock is given a fractional rate offset prior to launch of -4.465×10^{-10} from its nominal frequency of exactly 10.23 MHz, so that when in orbit its average rate is the same as the rate of a clock on the ground. The actual frequency of the satellite clock prior to launch is thus 10.229 999 995 43 MHz.

Although GPS satellite orbits are nominally circular, there is always some residual eccentricity. The eccentricity causes the orbit to be slightly elliptical. Thus, the velocity and gravitational potential vary slightly over one revolution and, although the principal secular effect is compensated by a rate offset, there remains a small residual variation that is proportional to the eccentricity. For example, with an orbital eccentricity of 0.02, there is a relativistic sinusoidal variation in the apparent clock time having an amplitude of 46 ns at the orbital period. This correction must be calculated and taken into account in the user's receiver.

The third relativistic effect is the Sagnac effect. For a stationary terrestrial receiver on the geoid, the Sagnac correction can be as large as 133 ns (corresponding to a GPS signal propagation time of 86 ms and a velocity of 465 m/s at the equator in the ECI frame). This correction is also applied in the receiver.

Higher-order effects not presently modeled in the GPS include the Earth oblateness contribution to the gravitational redshift, the tidal potentials of the Moon and Sun, and the effect of the gravitational potential on the speed of signal propagation. When satellite cross links are implemented in the future, the orbital eccentricity effect will have to be taken into account at both the transmitter and receiver.

OTHER SATELLITES IN EARTH ORBIT

To illustrate their orders of magnitude, the relativistic effects on clocks and signal propagation for a variety of Earth orbiting satellites are compared in Table 1.

Table 1. Relativistic effects on clocks and signals for satellites in Earth orbit.

Constants							
Velocity of light	m/s	299 792 458					
Gravitational constant of Earth	km^3/s^2	398 600.44					
Radius of Earth	km	6378.137					
J_2 oblateness coefficient		0.0010826					
Angular velocity of Earth rotation	rad/s	7.292×10^{-5}					
Geopotential on geoid U_0	m^2/s^2	6.264×10^{7}					
U_0/c^2		-6.969×10^{-10}					
Satellite orbital properties							
Satellite		**ISS**	**GLONASS**	**GPS**	**Galileo**	**Molniya**	**GEO**
Semimajor axis	km	6766	25510	26561.8	29994	26562	42164
Eccentricity		0.01	0.02	0.02	0.02	0.722	0.01
Inclination	deg	51.6	64.8	55.0	56.0	63.4	0.1
Argument of perigee	deg	0	0	0	0	250	0
Apogee altitude	km	456	19642	20715	24216	39362	36208
Perigee altitude	km	320	18622	19652	23016	1006	35364
Ascending node altitude	km	320	18622	19652	23016	10507	35364
Period of revolution	s	5539	40549	43082	51697	43083	86164
Mean motion	rev/d	15.6	2.1	2.0	1.7	2.0	1.0
Mean velocity	km/s	7.675	3.953	3.874	3.645	3.874	3.075
Clock effects							
Secular time dilation	µs/d	-28.2	-7.4	-7.1	-6.3	-7.1	-4.4
Secular redshift	µs/d	3.5	45.1	45.7	47.3	45.7	51.0
Net secular effect	µs/d	-24.7	37.7	38.6	41.1	38.6	46.6
Amplitude of periodic effect due to eccentricity	ns	12	45	46	49	1653	29
Secular oblateness contribution to redshift	ns/d	23.7	0.8	0.5	0.4	2.5	-0.1
Amplitude of periodic effect due to oblateness	ps	264	50	38	33	167	0
Amplitude of periodic tidal effect of Moon	ps	0.0	1.0	1.2	1.8	1.2	6.1
Amplitude of periodic tidal effect of Sun	ps	0.0	0.5	0.5	0.8	0.5	2.7
Signal propagation							
Maximum Sagnac effect	ns	13	131	136	155	234	218
Gravitational propagation delay along radius	ps	0.8	-3.5	-4.7	-9.1	-4.7	-27.3
Amplitude of periodic fractional Doppler shift	10^{-12}	13.1	7.0	6.7	5.9	241.1	2.1

RELATIVISTIC TRANSFORMATION FROM MARS TO EARTH

For clocks in communication and navigation systems used for space exploration, analogous corrections are required. Thus, appropriate relativistic transformations are necessary in transferring time from one frame of reference to another, for example between a clock on Mars to a clock on Earth. Reference data used in these calculations are summarized in Table 2.

Table 2. Reference data for the Sun, Earth, and Mars.

Mass		
	Sun	1.9891×10^{30} kg
	Earth	5.9742×10^{24} kg
	Mars	0.6419×10^{24} kg
Planetary radius		
	Earth	6378 km
	Mars	3397 km
Orbital semimajor axis		
	Earth	1.000 AU = 1.496×10^{8} km
	Mars	1.524 AU = 2.279×10^{8} km
Orbital period		
	Earth	365.2422 d
	Mars	686.9297 d
Average orbital velocity		
	Earth	29.8 km/s
	Mars	24.1 km/s
Orbital eccentricity		
	Earth	0.0167
	Mars	0.0934
Speed of light		299,792,458 m/s (exact)
Gravitational constant		$6.6726 \times 10^{-11} m^3 / kg\ s^2$

The analysis of time transfer must be carried out in a common coordinate system. A convenient coordinate system is one whose origin is at the solar system barycenter. The corresponding coordinate time is called Barycentric Coordinate Time (TCB). For time transfer between Mars and Earth, two transformations are required. The first transformation is from Terrestrial Time (TT) to Barycentric Coordinate Time (TCB). The second is from Barycentric Coordinate Time (TCB) to Mars Time (MT). The coordinate time TCB is an intermediate variable that ultimately cancels out. These transformations are illustrated schematically in Figure 1.

BARYCENTRIC COORDINATE TIME - TERRESTRIAL TIME

The elapsed coordinate time Δt in a barycentric coordinate system corresponding to the proper time $\Delta\tau$ maintained by a clock, having an arbitrary position and velocity in this coordinate system, is

$$\Delta t = \int_{\tau_0}^{\tau} \left(1 + \frac{1}{c^2} U(\mathbf{r}) + \frac{1}{2}\frac{1}{c^2} v^2 \right) d\tau$$

where **r** and **v** are the barycentric position and velocity of the clock and $U(\mathbf{r})$ is the gravitational potential of all the bodies in the solar system (including the Earth) evaluated at the clock. The integral depends on the position and velocity of the clock in the barycentric coordinate system. The coordinate time Δt is identified with TCB.

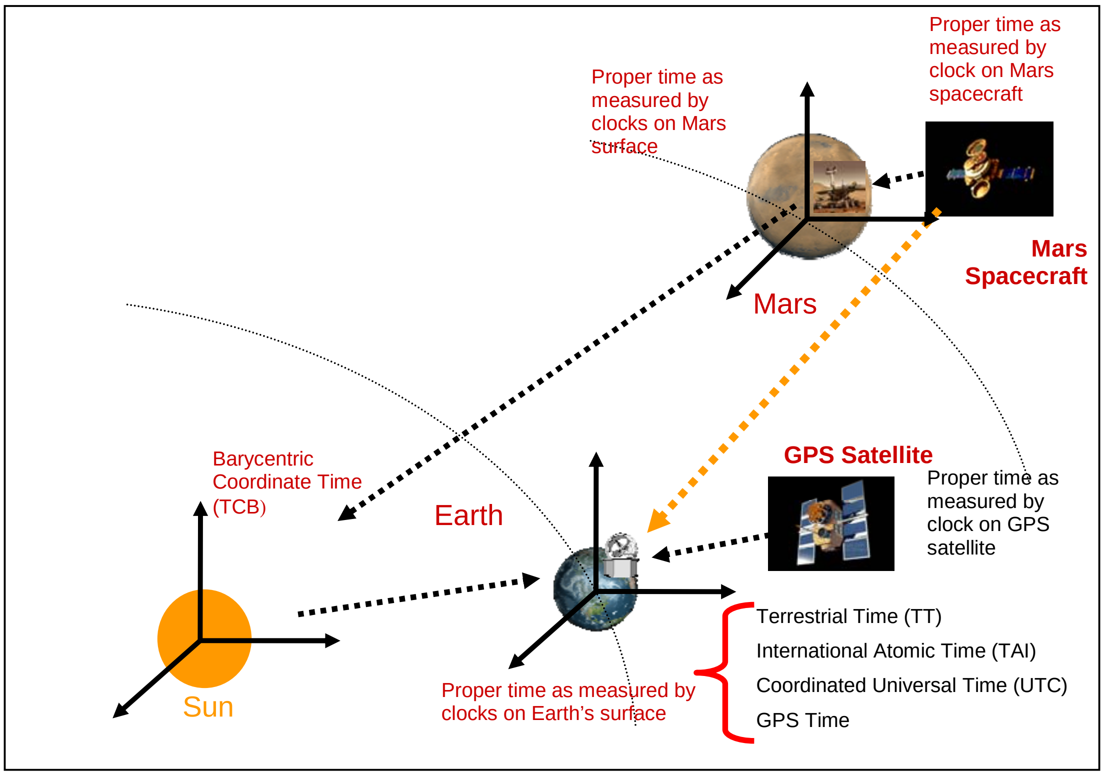

Figure 1. Relativistic time transformations.

It is desirable to separate the clock–dependent part from the clock–independent part. In this approximation, one may express **r** and **v** as $\mathbf{r} = \mathbf{r}_E + \mathbf{R}$ and $\mathbf{v} = \mathbf{v}_E + \dot{\mathbf{R}}$, where $\mathbf{r}_E$ and $\mathbf{v}_E$ are the barycentric position and velocity of the Earth's center of mass, and $\mathbf{R}$ and $\dot{\mathbf{R}}$ are the geocentric position and velocity of the clock, as illustrated in Figure 2. The total potential at position **r** is

$$U(\mathbf{r}) = U_E(\mathbf{r}) + U_{\text{ext}}(\mathbf{r})$$

where U_E is the Newtonian potential of the Earth and U_{ext} is the external Newtonian potential of all of the solar system bodies apart from the Earth. The external potential may be expressed

$$U_{\text{ext}}(\mathbf{r}) \approx U_{\text{ext}}(\mathbf{r}_E) + \nabla U_{\text{ext}} \cdot \mathbf{R}$$

Also,

$$v^2 = v_E^2 + 2\,\dot{\mathbf{R}}\cdot\mathbf{v}_E + |\,\dot{\mathbf{R}}\,|^2$$

and

$$\dot{\mathbf{R}}\cdot\mathbf{v}_E = \frac{d}{dt}(\mathbf{R}\cdot\mathbf{v}_E) - \mathbf{R}\cdot\mathbf{a}_E$$

where

$$\mathbf{a}_E \equiv \frac{d\mathbf{v}_E}{dt} = \nabla U_{\text{ext}}$$

is the Earth's acceleration in the barycentric coordinate system. Substituting these expressions into the integral, one obtains

$$\int_{\tau_0}^{\tau}\left\{1+\frac{1}{c^2}\left[U_E(\mathbf{r})+U_{ext}(\mathbf{r}_E)+\nabla U_{ext}\cdot\mathbf{R}\right]+\frac{1}{c^2}\left[\frac{1}{2}v_E{}^2+\frac{d}{dt}(\mathbf{R}\cdot\mathbf{v}_E)-\mathbf{R}\cdot\mathbf{a}_E+\frac{1}{2}|\,\dot{\mathbf{R}}\,|^2\right]\right\}\,d\tau\,.$$

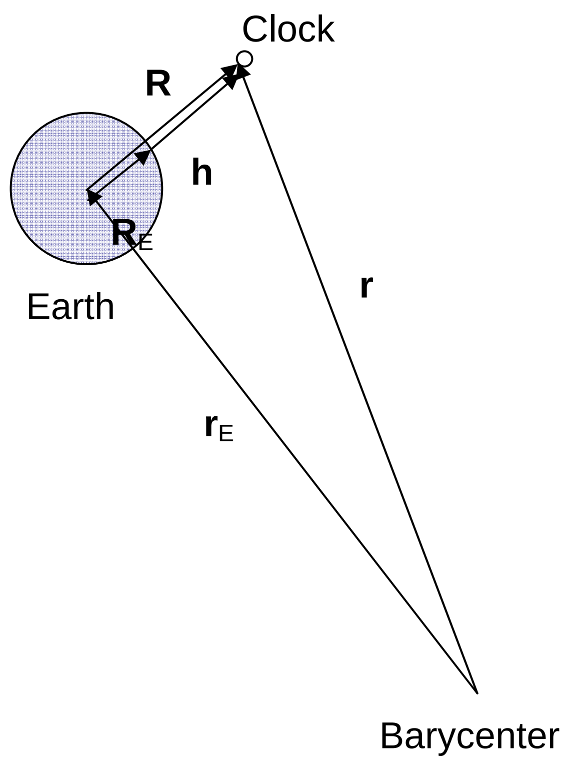

Figure 2. Geometry of clock, Earth, and solar system barycenter.

The gravitational acceleration terms $\nabla U_{ext}\cdot\mathbf{R}$ and $\mathbf{R}\cdot\mathbf{a}_E$ cancel out. Thus, as $U_E(\mathbf{r}) = U_E(\mathbf{R})$ (they refer to the same point), the elapsed coordinate time is

$$\Delta t \approx \Delta\tau + \frac{1}{c^2}\int_{t_0}^{t}\left(U_{\text{ext}}(\mathbf{r}_E)+\frac{1}{2}v_E^2\right)dt + \frac{1}{c^2}\int_{t_0}^{t}\left(U_E(\mathbf{R})+\frac{1}{2}|\,\dot{\mathbf{R}}\,|^2\right)dt + \frac{1}{c^2}\mathbf{R}\cdot\mathbf{v}_E\Big|_{t_0}^{t}$$

This equation is completely general, regardless of the position of the clock. The first term is the proper time measured by the clock. The second term is due to the combined redshift and time dilation effects at the geocenter with respect to the barycenter and is independent of the clock. The third term is the time difference between a clock at the geocenter and a clock at position **R** with respect to the geocenter. The fourth term depends on the clock's velocity and position. In the limit of flat space-time, it represents the special relativity clock synchronization correction in the moving geocentric frame when observed from the barycentric frame. The cancellation of the two acceleration terms is a manifestation of the Principle of Equivalence for a freely falling frame of reference. That is, the Earth constitutes a freely falling frame in its orbit about the Sun.

The coordinate time scale of Terrestrial Time (TT) is equivalent to the proper time kept by a hypothetical clock on the geoid. This timescale is related to International Atomic Time (TAI) by the equation

$$\mathrm{TT} = \mathrm{TAI} + 32.184\ \mathrm{s}$$

The constant offset represents the difference between Ephemeris Time (an obsolete Newtonian timescale used for astronomical ephemerides which has been superseded by TT) and TAI at the defining epoch of TAI on 1 January 1958. For an actual clock at rest at an elevation h above the geoid where the local acceleration of gravity is g, the relation between TT and the proper time reading $\Delta\tau$ of the clock is

$$\mathrm{TT} = \Delta t' = (1 - g\,h / c^2)\,\Delta\tau$$

The transformation from TT to Geocentric Coordinate Time (TCG) is

$$\mathrm{TCG} - \mathrm{TT} = (W_{0E} / c^2)\,\Delta T = L_G\,\Delta T$$

where W_{0E} is the Earth's geopotential, $L_G \equiv W_{0E} / c^2 = 6.969\ 290\ 134 \times 10^{-10} \cong 60.2$ μs/d, and ΔT is the time elapsed since 1 January 1977 0 h TAI (JD 2443144.5).

In the general equation above, the first integral may be calculated by numerical integration or it may be represented by an analytical formula. It is expressed as the sum of a secular term $L_C\,\Delta T$ and periodic terms P. For a clock on the geoid the second integral is simply $W_{0E}\,\Delta t$. Thus for a clock on the geoid

$$\begin{aligned}\mathrm{TCB} &= \mathrm{TT} + \frac{1}{c^2}\int_{t_0}^{t}\left(U_{\mathrm{ext}}(\mathbf{r}_E) + \frac{1}{2}v_E^2\right)dt + L_G\,\Delta T + \frac{1}{c^2}\mathbf{R}_E\cdot\mathbf{v}_E\Big|_{t_0}^{t} \\ &= \mathrm{TT} + L_C\,\Delta T + P + L_G\,\Delta T + \frac{1}{c^2}\mathbf{R}_E\cdot\mathbf{v}_E\Big|_{t_0}^{t}\end{aligned}$$

where $L_C = 1.480\ 826\ 867\ 41 \times 10^{-8} \cong 1.28$ ms/d. The diurnal term has a maximum amplitude of 2.1 μs (for a clock on the equator). The leading terms in the evaluation of the integral are

$$\frac{1}{c^2}\int_{t_0}^{t}\left(U_{E\,ext}(\mathbf{r}_E) + \frac{1}{2}v_E^2\right)dt \approx \frac{3}{2}\frac{1}{c^2}\frac{GM_S}{a_E}\Delta T + \frac{2}{c^2}\sqrt{GM_S\,a_E}\;e_E\,\sin E_E$$

where GM_S is the gravitational constant of the Sun, and where a_E and e_E are the Earth's orbital semimajor axis and eccentricity. The first term is an approximation to $L_C\,\Delta T$. The second term is the principal periodic term in P, which has amplitude of 1.7 ms.

BARYCENTRIC COORDINATE TIME - MARS TIME

By an analogous derivation, one finds that the transformation from Barycentric Coordinate Time (TCB) to Mars Time (MT) is given by

$$\begin{aligned}\mathrm{TCB} &= \mathrm{MT} + \frac{1}{c^2}\int_{t_0}^{t}\left(U_{\mathrm{ext}}\left(\mathbf{r}_M\right) + \frac{1}{2}v_M^2\right)dt + L_M\,\Delta T + \frac{1}{c^2}\mathbf{R}_M\cdot\mathbf{v}_M\Big|_{t_0}^{t}\\ &= \mathrm{MT} + L_{CM}\,\Delta T + P + L_M\,\Delta T + \frac{1}{c^2}\mathbf{R}_M\cdot\mathbf{v}_M\Big|_{t_0}^{t}\end{aligned}$$

where $L_{CM} = 0.972\times10^{-8} \cong 0.84$ ms/d, $L_M \equiv W_{0M}\ /\ c^2 = 1.403\times10^{-10} \cong 12.1$ µs/d, W_{0M} is the areopotential ("geopotential" on Mars), P represents periodic terms, and $\mathbf{R}_M$ is the areocentric position of the clock on the surface of Mars. The diurnal term has a maximum amplitude of 0.9 µs. The leading terms in the integral are

$$\frac{1}{c^2}\int_{t_0}^{t}\left(U_{M\ ext}\left(\mathbf{r}_M\right) + \frac{1}{2}v_M^2\right)dt \approx \frac{3}{2}\frac{1}{c^2}\frac{GM_S}{a_M}\Delta T + \frac{2}{c^2}\sqrt{GM_S\ a_M}\ e_M\ \sin E_M$$

where GM_S is the gravitational constant of the Sun, and where a_M and e_M are the Mars orbital semimajor axis and eccentricity. The first term is an approximation to $L_{CM}\,\Delta T$. The second term is the principal periodic term in P, which has amplitude of 11.4 ms.

NET EFFECTS: MARS TIME - TERRESTRIAL TIME

The results of these calculations are summarized in Table 3. The difference in the readings of a clock on the surface of Mars and a clock on the surface of the Earth has both secular and periodic terms. The difference between Mars Time (MT) and Terrestrial Time (TT) is

$$\mathrm{MT} - \mathrm{TT} = (\mathrm{TCB} - \mathrm{TT}) - (\mathrm{TCB} - \mathrm{MT})$$

The net secular drift is (1.28 ms/d + 0.06 ms/d) – (0.84 ms/d + 0.01 ms/d) = 0.49 ms/d. The amplitudes of the periodic variations are: (*a*) 1.7 ms at the Earth orbital period (365.2422 d); (*b*) 11.4 ms at the Mars orbital period (687 d). Therefore, in the transfer of time between a clock on Mars and a clock on the Earth, there are both secular and periodic effects that are on the order of 1 to 10 milliseconds. For a navigation ranging system referenced to a clock on Mars and an ephemeris referenced to clocks on Earth, the radial position error could be as much as 3,000 km if the relativistic effects were not modeled.

Table 3. Principal relativity effects for time transfer between Mars and Earth.

Geoid to Geocenter	
Secular drift	60.2 µs/d
Maximum amplitude of diurnal term	2.1 µs
Geocenter to Barycenter	
Secular drift	1.28 ms/d
Amplitude of principal periodic term	1.7 ms
Mars surface to Mars center	
Secular drift	12.1 µs/d
Maximum amplitude of diurnal term	0.9 µs
Mars center to Barycenter	
Secular drift	0.84 ms/d
Amplitude of principal periodic term	11.4 ms

RELATIVISTIC TRANSFORMATION FROM THE MOON TO EARTH

For time transfer from the surface of the Moon to the surface of the Earth, the procedure is similar, but the relative magnitudes of the terms are different. A convenient coordinate system is one whose origin is at the center of the Earth. (The motion of the Earth's center about the center of mass of the Earth-Moon system will be neglected.) As above, the difference between Geocentric Coordinate Time and Terrestrial Time is

$$\mathrm{TCG} - \mathrm{TT} = L_G \, \Delta T$$

where L_G = 60.2 µs/d. But in addition, TCG is related to Lunar Time (LT), the proper time measured by clocks on the Moon's surface, by the equation

$$\mathrm{TCG} = \mathrm{LT} + L_{Cm} \, \Delta T + P + L_m \, \Delta T + \frac{1}{c^2} \mathbf{R}_m \cdot \mathbf{v}_m \Big|_{t_0}^{t}$$

where the subscript *m* refers to the Moon and

$$L_{Cm} \approx \frac{3}{2} \frac{1}{c^2} \frac{GM_E}{a_m} \quad ,$$

$$P \approx \frac{2}{c^2} \sqrt{GM_E \, a_m} \; e_m \; \sin E_m \quad ,$$

$$L_m \approx \frac{1}{c^2} \frac{GM_m}{R_m} \quad .$$

Reference data for the Moon are summarized in Table 4.

Table 4. Reference data for the Moon.

Mass	0.07353×10^{24} kg
Radius	1738.2 km
Orbital semimajor axis	384,400 km
Orbital eccentricity	0.05490
Average orbital velocity	1.023 km/s
Distance of geocenter from barycenter	4671 km

Thus $L_{Cm} = 1.731 \times 10^{-11} = 1.5$ μs/d, $(2/c^2)\sqrt{GM_E\, a_m}\ e_m = 0.48\ \mu\mathrm{s}$, and $L_m = 3.141 \times 10^{-11} = 2.7$ μs/d. The difference between Lunar Time (LT) and Terrestrial Time (TT) is

$$\mathrm{LT} - \mathrm{TT} = (\mathrm{TCG} - \mathrm{TT}) - (\mathrm{TCG} - \mathrm{LT})$$

The net secular drift rate is 60.2 μs/d – (1.5 μs/d + 2.7 μs/d) = 56.0 μs/d and the amplitude of the periodic effect is 0.48 μs at the Moon's orbital period (27.3 d).

TIME EPHEMERIS AND THE PLANETARY EPHEMERIDES

Another coordinate time of interest is Time Ephemeris t_{eph}, which is a solar system barycenter coordinate time that has been rescaled to have the same secular rate as TT. This relativistic coordinate time is used in the planetary and satellite ephemerides published by the Jet Propulsion Laboratory. (Although it has a similar name, it is not related to the Newtonian scale of Ephemeris Time.) This definition of time changes the secular rates of coordinate time by subtracting the quantity $L_G + L_C = 1.551 \times 10^{-8} = 1.34$ ms/day from the given secular rates. Using the recent DE410 ephemeris published by JPL, numerical results of the difference between proper time and Time Ephemeris for a Mars lander is shown in Figure 3. The simulations are for 6 years and begin on the J2000 epoch of 1 January 2000. The results clearly show a large 11.4 ms periodic term for a clock on Mars. The results also show that relativistic time calculations at Mars must account for the secular rate difference between *in-situ* clocks and Earth clocks.

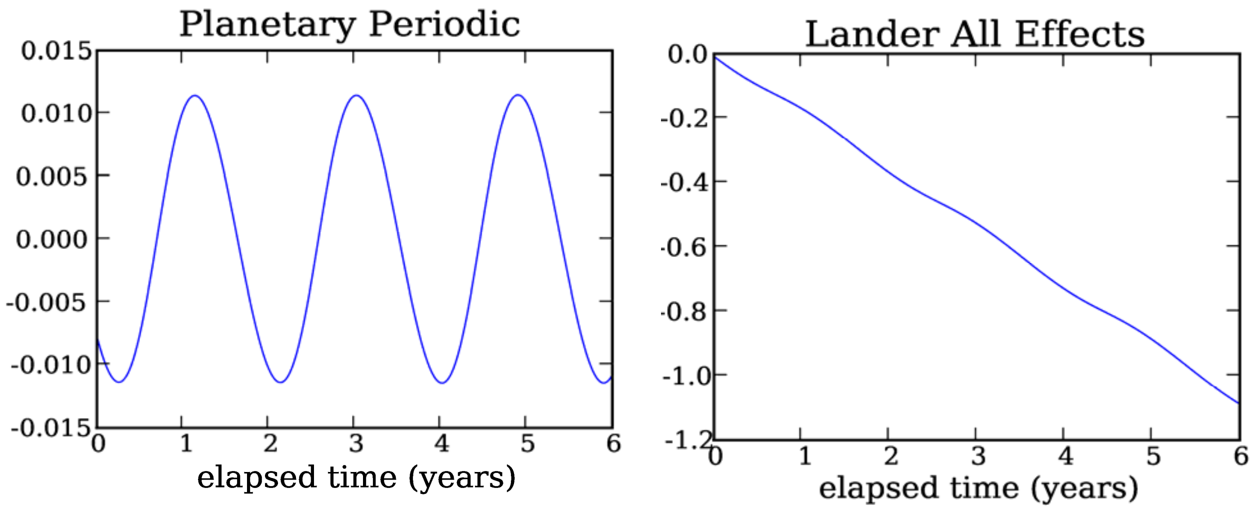

Figure 3. Secular and periodic relativistic effects, proper time – coordinate time (seconds), for a Mars lander using Time Ephemeris as the coordinate time.

CONCLUSION

Transformations between clocks operating on the Earth and clocks at Mars or on the Moon will become an essential activity for future space missions. The analysis of this paper has shown that the relativistic effects at Mars are comprised of a secular rate difference of about 0.49 ms/d and periodic variations with amplitudes of 1.7 ms and 11.4 ms relative to Earth-based clocks. The secular effect for the Moon is about an order of magnitude less. Accurate time transfer in the solar system for communications and navigation systems requires the consideration of these relativistic effects.

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